

Movable Intersection And Bigness Criterion

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Abstract

In this note, we give a Morse-type bigness criterion for the difference of two pseudo-effective $(1, 1)$ -classes by using movable intersections. And with this result we give a Morse-type bigness criterion for the difference of two movable $(n - 1, n - 1)$ -classes.

1 Introduction

Let X be a smooth projective variety of dimension n , and let A, B be two nef line bundles over X . Then we have the fundamental inequality

$$\mathrm{vol}(A - B) \geq A^n - nA^{n-1} \cdot B,$$

which is first discovered as a consequence of Demailly's holomorphic Morse inequalities (see [Dem85], [Siu93], [Tra95]). Thus the above inequality is usually called algebraic Morse inequality for line bundles. Recall that the volume of a holomorphic line bundle L is defined as

$$\mathrm{vol}(L) := \limsup_{k \rightarrow +\infty} \frac{n!}{k^n} h^0(X, \mathcal{O}(kL)).$$

And L is called a big line bundle if $\mathrm{vol}(L) > 0$. In particular, the Morse-type inequality for A, B implies $A - B$ must be a big line bundle if $A^n - nA^{n-1} \cdot B > 0$. This provides a very effective way to construct holomorphic sections; see [DMR10, Dem11] for related applications.

Assume L is a holomorphic line bundle over a compact Kähler manifold X , then it is proved by [Bou02b, Theorem 1.2] that the volume of L can be characterized as the maximum of the Monge-Ampère mass of the positive curvature currents contained in the class $c_1(L)$. This naturally extends the volume function vol to transcendental $(1, 1)$ -classes over compact complex manifold (see [Bou02b, Definition 1.3] or [BDPP13, Definition 3.2]).

Recall that Demailly's conjecture on (weak) transcendental holomorphic Morse inequality over compact Kähler manifolds is stated as following.

Conjecture 1.1. (see [BDPP13, Conjecture 10.1]) Let X be a compact Kähler manifold of dimension n , and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two nef classes. Then we have

$$\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

In particular, if $\alpha^n - n\alpha^{n-1} \cdot \beta > 0$ then there exists a Kähler current in the class $\alpha - \beta$.

In our previous work [Xia13], we proved that if $\alpha^n - 4n\alpha^{n-1} \cdot \beta > 0$ then there exists a Kähler current in the class $\alpha - \beta$. Recently, by keeping the same method which goes back to [Chi13] and with the new technique introduced by [Pop14], [Pop14] proved that the constant $4n$ can be improved to be the natural and optimal constant n . Thus we have a Morse-type criterion for the difference of two transcendental nef classes – indeed, our results in this note depends mainly on this important improvement.

It is natural to ask whether the above Morse-type bigness criterion “ $\alpha^n - n\alpha^{n-1} \cdot \beta > 0 \Rightarrow \text{vol}(\alpha - \beta) > 0$ ” for nef classes can be generalized to pseudo-effective $(1, 1)$ -classes. Towards this generalization, we need the movable intersection products (denoted by $\langle - \rangle$) of pseudo-effective $(1, 1)$ -classes developed in [Bou02a, BDPP13]. Then our problem can be stated as following:

Let X be a compact Kähler manifold of dimension n , and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Does $\text{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$ imply there exists a Kähler current in the class $\alpha - \beta$?

Unfortunately, a very simple example due to [Tra95] implies the above generalization does not always hold.

Example 1.1. (see [Tra95, Example 3.8]) Let $\pi : X \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 along a point p . Let $R = \pi^*H$, where H is the hyperplane line bundle on \mathbb{P}^2 . Let $E = \pi^{-1}(p)$ be the exceptional divisor. Then for every positive integral k , the space of global holomorphic sections of $k(R - 2E)$ is the space of homogeneous polynomials in three variables of degree at most k and vanishes of order $2k$ at p ; hence $k(R - 2E)$ does not have any global holomorphic sections. The space $H^0(X, \mathcal{O}(k(R - 2E))) = \{0\}$ implies $R - 2E$ can not be big. However, we have $R^2 - R \cdot 2E > 0$ as $R^2 = 1$ and $R \cdot E = 0$.

As the first result of this note, we show it holds if β is movable. Here β being movable means the negative part of β vanishes in its divisorial Zariski decomposition. In particular, if $\beta = c_1(L)$ for some pseudo-effective line bundle, then β being movable is equivalent to that the base locus of $mL + A$ is of codimension at least two for a fixed ample line bundle A and for large m .

Theorem 1.1. Let X be a compact Kähler manifold of dimension n , and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes with β movable. Then $\text{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$ implies there exists a Kähler current in the class $\alpha - \beta$.

Remark 1.1. In the case when $\beta = 0$, Theorem 1.1 recovers [Bou02b, Theorem 4.7], and when α is also nef, it is [DP04, Theorem 0.5].

An ancillary goal of the note is to explain the fact that Demailly’s conjecture on weak transcendental holomorphic Morse inequality over compact Kähler manifolds is equivalent to the \mathcal{C}^1 differentiability of the volume function for transcendental $(1, 1)$ -classes. Though not stated explicitly, this fact is already known by [BFJ09] and the key ingredients are implicitly contained in [BFJ09, BDPP13].

Proposition 1.1. Let X be a compact Kähler manifold of dimension n . Then the following statements are equivalent:

1. Let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two nef classes, then we have

$$\text{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta.$$

2. Let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes with β movable, then

$$\text{vol}(\alpha - \beta) \geq \langle \alpha^n \rangle - n\langle \alpha^{n-1} \rangle \cdot \beta.$$

3. Let $\alpha, \gamma \in H^{1,1}(X, \mathbb{R})$ be two $(1, 1)$ -classes with α big, then we have

$$\left. \frac{d}{dt} \right|_{t=0} \text{vol}(\alpha + t\gamma) = n\langle \alpha^{n-1} \rangle \cdot \gamma.$$

As an application of Proposition 1.1 and the \mathcal{C}^1 differentiability of the volume function for line bundles (see [BFJ09, Theorem A]), the algebraic Morse inequality can be generalized as following. It covers the previous result [Tra11, Corollary 3.2].

Theorem 1.2. Let X be a smooth projective variety of dimension n , and let α, β be the first Chern classes of two pseudo-effective line bundles with β movable. Then we have

$$\text{vol}(\alpha - \beta) \geq \text{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta.$$

Remark 1.2. In particular, if α is nef and β is movable then we have $\text{vol}(\alpha - \beta) \geq \alpha^n - n\alpha^{n-1} \cdot \beta$ which is just [Tra11, Corollary 3.2].

Finally, as an application of Theorem 1.1, we give a Morse-type bigness criterion for the difference of two movable $(n-1, n-1)$ -classes.

Theorem 1.3. Let X be a compact Kähler manifold of dimension n , and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two pseudo-effective classes. Then $\text{vol}(\alpha) - n\alpha \cdot \langle \beta^{n-1} \rangle > 0$ implies there exists a strictly positive $(n-1, n-1)$ -current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$.

2 Technical preliminaries

2.1 Resolution of singularities of positive currents

Let X be a compact complex manifold, and let T be a d -closed almost positive $(1, 1)$ -current on X , that is, there exists a smooth $(1, 1)$ -form γ such that $T \geq \gamma$. If $\gamma = 0$, then T is called a positive $(1, 1)$ -current and the class $\{T\}$ is called pseudo-effective; And if γ is a hermitian metric, then T is called a Kähler current and the class $\{T\}$ is called big; see [Dem12] for the basic theory of positive currents.

Demailly's regularization theorem (see [Dem92]) implies that we can always approximate the almost positive $(1, 1)$ -current T by a family of almost positive closed $(1, 1)$ -currents T_k with analytic singularities such that $T_k \geq \gamma - \varepsilon_k \omega$, where $\varepsilon_k \downarrow 0$ is a sequence of positive constants and ω is a fixed hermitian metric. In particular, when T is a Kähler current, it can be approximated by a family of Kähler currents with analytic singularities.

When T has analytic singularities along an analytic subvariety $V(\mathcal{I})$ where $\mathcal{I} \subset \mathcal{O}_X$ is a coherent ideal sheaf, by blowing up along $V(\mathcal{I})$ and then resolving the singularities, we get a modification $\mu : \tilde{X} \rightarrow X$ such that $\mu^*T = \tilde{\theta} + [D]$ where $\tilde{\theta}$ is an almost positive smooth $(1,1)$ -form with $\tilde{\theta} \geq \mu^*\gamma$ and D is an effective \mathbb{R} -divisor; see e.g. [BDPP13, Theorem 3.1]. In particular, if T is positive, then $\tilde{\theta}$ is a smooth positive $(1,1)$ -form. We call such a modification the log-resolution of singularities of T .

For almost positive $(1,1)$ -current T , we can always decompose T with respect to the Lebesgue measure; see e.g. [Bou02b, Section 2.3]. We write $T = T_{ac} + T_{sg}$ where T_{ac} is the absolutely continuous part and T_{sg} is the singular part. The absolutely part T_{ac} can be seen as a form with L^1_{loc} coefficients, and the wedge product $T_{ac}^k(x)$ makes sense for almost every point x . We always have $T_{ac} \geq \gamma$ since γ is smooth. If T has analytic singularities along V , then $T_{ac} = \mathbf{1}_{X \setminus V}T$. However, in general T_{ac} is not closed even if T is closed. We have the following proposition.

Proposition 2.1. Let T_1, \dots, T_k be k almost positive closed $(1,1)$ -currents with analytic singularities on X and let ψ be a smooth $(n-k, n-k)$ -form. Let $\mu : \tilde{X} \rightarrow X$ be a simultaneous log-resolution with $\mu^*T_i = \tilde{\theta}_i + [D_i]$. Then

$$\int_X T_{1,ac} \wedge \dots \wedge T_{k,ac} \wedge \psi = \int_{\tilde{X}} \tilde{\theta}_1 \wedge \dots \wedge \tilde{\theta}_k \wedge \mu^*\psi.$$

Proof. This is obvious since μ is an isomorphism outside a proper analytic subvariety and $T_{1,ac} \wedge \dots \wedge T_{k,ac}$ puts no mass on such subset and $\tilde{\theta}_i$ is smooth on \tilde{X} . \square

2.2 Movable cohomology classes

We first briefly recall the definition of divisorial Zariski decomposition and the definition of movable $(1,1)$ -class on compact complex manifold; see [Bou04], see also [Nak04] for the algebraic approach.

Let X be a compact complex manifold of dimension n and let α be a pseudo-effective $(1,1)$ -class over X , then one can always associate an effective divisor $N(\alpha) := \sum \nu(\alpha, D)D$ to α where the sum ranges among all prime divisors on X . The class $\{N(\alpha)\}$ is called the negative part of α . And $Z(\alpha) = \alpha - \{N(\alpha)\}$ is called the positive part of α . The decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ then is the divisorial Zariski decomposition of α .

Definition 2.1. Let X be a compact complex manifold of dimension n , and let α be a pseudo-effective $(1,1)$ -class. Then α is called movable if $\alpha = Z(\alpha)$.

Proposition 2.2. (see [Bou04, Proposition 2.3]) Let α be a movable $(1,1)$ -class and let ω be a Kähler class, then for any $\delta > 0$ there exist a modification $\mu : Y \rightarrow X$ and a Kähler class $\tilde{\omega}$ over Y such that $\alpha + \delta\omega = \mu_*\tilde{\omega}$.

Remark 2.1. In [Bou04], α is called modified nef if $\alpha = Z(\alpha)$ (see [Bou04, Definition 2.2 and Proposition 3.8]). Here we call it movable in order to keep the same notation as the algebraic geometry situation. Let L be a big line bundle over a smooth projective variety and let $\alpha = c_1(L)$. Then α is modified nef if and only if L is movable, that is, its base locus is of codimension at least two.

Inspired by [BDPP13, Definition 1.3, Theorem 1.5 and Conjecture 2.3], the definition of movable $(n-1, n-1)$ -classes in the Kähler setting can be formulated as following.

Definition 2.2. Let X be a compact Kähler manifold of dimension n , and let $\gamma \in H^{n-1, n-1}(X, \mathbb{R})$. Then γ is called a movable $(n-1, n-1)$ -class if it is in the closure of the convex cone generated by cohomology classes of the form $\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle$ with every α_i pseudo-effective.

Remark 2.2. When X is a smooth projective variety of dimension n , [BDPP13, Theorem 1.5] implies the rational movable $(n-1, n-1)$ -classes are the same with the classes of movable curves.

2.3 Movable intersections

In this section, we take the opportunity to briefly explain the well known fact that the several definitions of movable intersections of pseudo-effective $(1, 1)$ -classes over compact Kähler manifold coincide; see [Bou02a, BDPP13, BEGZ10] for the analytic constructions over compact Kähler manifold and [BFJ09] for the algebraic construction on smooth projective variety. We remark that it is helpful to know the definition of movable intersections of pseudo-effective $(1, 1)$ -classes can be interpreted in several equivalent ways.

Let $\alpha_1, \dots, \alpha_k \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes on a compact Kähler manifold of dimension n . By the common basic property of these definitions of movable intersection products, we only need to verify the respectively defined positive (k, k) cohomology classes $\langle \alpha_1 \cdot \dots \cdot \alpha_k \rangle$ coincide when all the classes are big. Firstly, by the definition of Riemann-Zariski space, it is clear from [BDPP13, Theorem 3.5] and [BFJ09, Definition 2.5] that the two definitions of movable intersection products are the same when X is a smooth projective variety defined over \mathbb{C} and all the classes α_i are in the Néron-Severi space. Next, by testing on $\partial\bar{\partial}$ -closed smooth positive $(n-k, n-k)$ -forms, [BDPP13, Theorem 3.5], [BEGZ10, Definition 1.17, Proposition 1.18 and Proposition 1.20] and [Bou02a, Definition 3.2.1 and Lemma 3.2.5] imply these three definitions give the same positive cohomology class in $H^{k,k}(X, \mathbb{R})$; see [Pri13, Proposition 1.10] for the detailed proof.

3 Main results

Now let us begin to prove our main results. We first give a Morse-type bigness criterion for the difference of two pseudo-effective $(1, 1)$ -classes by using movable intersections. To this end, we need some properties of movable intersections.

Proposition 3.1. Let X be a compact Kähler manifold of dimension n , and let $\alpha_1, \dots, \alpha_k \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes. Let $\mu : Y \rightarrow X$ be a modification with Y Kähler, then we have

$$\mu^* \langle \alpha_1 \cdot \dots \cdot \alpha_k \rangle = \langle \mu^* \alpha_1 \cdot \dots \cdot \mu^* \alpha_k \rangle.$$

Proof. By taking limits, we only need to verify the case when all the classes α_i are big. By [BEGZ10, Definition 1.17], the movable intersections can be defined by positive currents of minimal singularities, that is,

$$\langle \alpha_1 \cdot \dots \cdot \alpha_k \rangle := \{ \langle T_{1, \min} \wedge \dots \wedge T_{k, \min} \rangle \},$$

where $\langle T_{1,\min} \wedge \dots \wedge T_{k,\min} \rangle$ is the non-pluripolar product of positive currents and $T_{i,\min}$ is a positive current in the big class α_i with minimal singularities. And if $\alpha_1, \dots, \alpha_k$ are merely pseudo-effective, we set

$$\langle \alpha_1 \cdot \dots \cdot \alpha_k \rangle := \lim_{\epsilon \rightarrow 0} \langle (\alpha_1 + \epsilon \omega) \cdot \dots \cdot (\alpha_k + \epsilon \omega) \rangle$$

where ω is an arbitrary Kähler class on X .

To prove the desired equality, using Poincaré duality, we need to verify

$$\mu^* \langle \alpha_1 \cdot \dots \cdot \alpha_k \rangle \cdot \{\eta\} = \langle \mu^* \alpha_1 \cdot \dots \cdot \mu^* \alpha_k \rangle \cdot \{\eta\}$$

for an arbitrary d -closed smooth $(n-k, n-k)$ -form η . Let $T_{i,\min} \in \alpha_i$ be a positive current with minimal singularities, then [BEGZ10, Proposition 1.12] implies $\mu^* T_{i,\min} \in \mu^* \alpha_i$ is also a positive current with minimal singularities. Thus we have

$$\langle \mu^* \alpha_1 \cdot \dots \cdot \mu^* \alpha_k \rangle = \{ \langle \mu^* T_{1,\min} \wedge \dots \wedge \mu^* T_{k,\min} \rangle \}.$$

By the definition of non-pluripolar products of d -closed positive $(1,1)$ -currents, these products do not put mass on pluripolar subsets. In particular, they do not put mass on proper analytic subvarieties. Indeed, by Demailly's regularization theorem, there exists an analytic Zariski open set where μ is an isomorphism and all the currents $T_{i,\min}$ are of locally bounded potentials. Integrating over this set, we get

$$\int_Y \mu^* \langle T_{1,\min} \wedge \dots \wedge T_{k,\min} \rangle \wedge \eta = \int_Y \langle \mu^* T_{1,\min} \wedge \dots \wedge \mu^* T_{k,\min} \rangle \wedge \eta.$$

Since η is arbitrary, this proves the equality $\mu^* \langle \alpha_1 \cdot \dots \cdot \alpha_k \rangle = \langle \mu^* \alpha_1 \cdot \dots \cdot \mu^* \alpha_k \rangle$. \square

Corollary 3.1. Let X be a compact Kähler manifold of dimension n , and let $\alpha_1, \dots, \alpha_{n-1}, \beta \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes with β nef. Then we have

$$\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \beta \rangle = \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \beta.$$

Proof. By taking limits, we can assume $\alpha_1, \dots, \alpha_{n-1}$ are big and β is Kähler.

First, by [BDPP13, Theorem 3.5], there exists a sequence of simultaneous log-resolutions $\mu_m : X_m \rightarrow X$ with $\mu_m^* \alpha_i = \omega_{i,m} + [D_{i,m}]$ and $\mu_m^* \beta = \gamma_m + [E_m]$ such that

$$\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \beta \rangle = \limsup_{m \rightarrow \infty} (\omega_{1,m} \cdot \dots \cdot \omega_{n-1,m} \cdot \gamma_m).$$

By the definition of $\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle$, we always have

$$\limsup_{m \rightarrow \infty} (\mu_m)_* (\omega_{1,m} \cdot \dots \cdot \omega_{n-1,m}) \leq \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle$$

in the sense of integrating against smooth $\partial\bar{\partial}$ -closed positive $(1,1)$ -forms. In particular, since β can be represented by a Kähler metric and $\mu_m^* \beta = \gamma_m + [E_m]$, we get

$$\begin{aligned} \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \beta \rangle &\leq \limsup_{m \rightarrow \infty} (\mu_m)_* (\omega_{1,m} \cdot \dots \cdot \omega_{n-1,m}) \cdot \beta \\ &\leq \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \beta. \end{aligned}$$

On the other hand, we claim that

$$\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \beta \rangle \geq \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \beta$$

if we merely assume β is movable. Without loss of generality, we can assume that $\beta = \pi_* \tilde{\omega}$ for some modification $\pi : Y \rightarrow X$ and some Kähler class $\tilde{\omega}$ on Y . Let $T_{i,\min} \in \alpha_i$ be the positive current with minimal singularities, and denote a Kähler metric in the Kähler class $\tilde{\omega}$ by the same symbol $\tilde{\omega}$. By Proposition 3.1 we have

$$\begin{aligned} \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \pi_* \tilde{\omega} &= \langle \pi^* \alpha_1 \cdot \dots \cdot \pi^* \alpha_{n-1} \rangle \cdot \tilde{\omega} \\ &= \int_Y \langle \pi^* T_{1,\min} \cdot \dots \cdot \pi^* T_{n-1,\min} \rangle \wedge \tilde{\omega} \\ &= \int_X \langle T_{1,\min} \cdot \dots \cdot T_{n-1,\min} \wedge \pi_* \tilde{\omega} \rangle \\ &\leq \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \pi_* \tilde{\omega} \rangle, \end{aligned}$$

where the third line follows by integrating $\langle \pi^* T_{1,\min} \cdot \dots \cdot \pi^* T_{n-1,\min} \rangle \wedge \tilde{\omega}$ outside a pluripolar subset (including the center of π) and the last line follows from the definition of $\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \pi_* \tilde{\omega} \rangle$ and [BEGZ10, Proposition 1.20].

In conclusion, if β is nef then we have the desired equality

$$\langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \cdot \beta \rangle = \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \beta.$$

□

Remark 3.1. It is not clear if Corollary 3.1 is true if β is merely movable.

Corollary 3.2. Let X be a compact Kähler manifold of dimension n . Let $\alpha'_1, \alpha_1, \dots, \alpha_{n-1}$ and β be pseudo-effective $(1,1)$ -classes such that $\alpha'_1 - \alpha_1$ is pseudo-effective and β is movable, then we have

$$\langle \alpha'_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \beta \geq \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot \beta.$$

Proof. Fix a Kähler class ω . By taking limits, we only need to verify

$$\langle \alpha'_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot (\beta + \delta\omega) \geq \langle \alpha_1 \cdot \dots \cdot \alpha_{n-1} \rangle \cdot (\beta + \delta\omega)$$

for any $\delta > 0$. Note that as β is movable there exists some modification $\mu : Y \rightarrow X$ and some Kähler class $\hat{\omega}$ such that $\mu_* \hat{\omega} = \beta + \delta\omega$. Then the result follows directly from Proposition 3.1 and Corollary 3.1. □

3.1 Theorem 1.1

Now we can give the proof of Theorem 1.1.

Proof. Fix a Kähler metric ω on X , and denote the Kähler class by the same symbol. By continuity and the definition of movable intersections, we have

$$\lim_{\delta \rightarrow 0} \langle (\alpha + \delta\omega)^n \rangle - n \langle (\alpha + \delta\omega)^{n-1} \rangle \cdot (\beta + \delta\omega) = \langle \alpha^n \rangle - n \langle \alpha^{n-1} \rangle \cdot \beta.$$

So $\langle(\alpha + \delta\omega)^n\rangle - n\langle(\alpha + \delta\omega)^{n-1}\rangle \cdot (\beta + \delta\omega) > 0$ for small $\delta > 0$. Note also that $\alpha - \beta = (\alpha + \delta\omega) - (\beta + \delta\omega)$. Thus to prove the bigness of the class $\alpha - \beta$, we can assume α is big, and assume $\beta = \mu_*\tilde{\omega}$ for some modification $\mu : Y \rightarrow X$ and some Kähler class $\tilde{\omega}$ on Y at the beginning.

By Proposition 3.1 and Corollary 3.1, our assumption then implies

$$\langle(\mu^*\alpha)^n\rangle - n\langle(\mu^*\alpha)^{n-1} \cdot \tilde{\omega}\rangle > 0.$$

We claim that this implies there exists a Kähler current in the class $\mu^*\alpha - \tilde{\omega}$, which then implies the bigness of the class $\alpha - \beta = \mu_*(\mu^*\alpha - \tilde{\omega})$.

Now it is reduced to prove the case when α is big and β is Kähler. Let ω be a Kähler metric in the class β . The definition of movable intersections implies there exists some Kähler current $T \in \alpha$ with analytic singularities along some subvariety V such that

$$\int_{X \setminus V} T^n - n \int_{X \setminus V} T^{n-1} \wedge \omega > 0.$$

Let $\pi : Z \rightarrow X$ be the log-resolution of the current T with $\pi^*T = \theta + [D]$ such that θ is a smooth positive $(1,1)$ -form on Z . By Proposition 2.1 we have

$$\int_Z \theta^n - n \int_Z \theta^{n-1} \wedge \pi^*\omega > 0.$$

The result of [Pop14] then implies there exists a Kähler current in the class $\{\theta - \pi^*\omega\}$. As $\pi^*\alpha = \{\theta + [D]\}$, this proves the bigness of the class $\alpha - \beta$.

Thus we finish the proof that there exists a Kähler current in the general case when α is pseudo-effective and β is movable. \square

Remark 3.2. By the proof of Corollary 3.1, we know $\langle\alpha^{n-1}\rangle \cdot \beta \leq \langle\alpha^{n-1} \cdot \beta\rangle$ if β is movable. So we have

$$\langle\alpha^n\rangle - n\langle\alpha^{n-1}\rangle \cdot \beta \geq \langle\alpha^n\rangle - n\langle\alpha^{n-1} \cdot \beta\rangle.$$

In particular, since $\langle\alpha^n\rangle$ and $\langle\alpha^{n-1} \cdot \beta\rangle$ depends only on the positive parts of α, β (see [Bou02a, Proposition 3.2.10]), we get the following weaker bigness criterion:

$$\langle\alpha^n\rangle - n\langle\alpha^{n-1} \cdot \beta\rangle > 0 \Rightarrow \text{vol}(Z(\alpha) - Z(\beta)) > 0.$$

In the case of Example 1.1, since R is nef and E is exceptional, we have $\langle R^2 \rangle - 2\langle R \cdot 2E \rangle = R^2 > 0$. We then get the bigness of $Z(R) - Z(2E) = R$.

3.2 Theorem 1.2

The algebraic Morse inequality tells us if L and F are two nef line bundles, then

$$\text{vol}(L - F) \geq L^n - nL^{n-1} \cdot F.$$

Recently, [Tra11] generalizes this result to the case when F is only movable. Assume that L is nef and F is pseudo-effective, and let $F = Z(F) + N(F)$ be the divisorial Zariski decomposition of F . Then [Tra11, Corollary 3.2] shows that

$$\mathrm{vol}(L - Z(F)) \geq L^n - nL^{n-1} \cdot Z(F).$$

Moreover, if we write the negative part $N(F) = \sum_j \nu_j D_j$ where $\nu_j > 0$ and let u be a nef class on X such that $c_1(\mathcal{O}_{TX}(1)) + \pi^*u$ is a nef class on $\mathbb{P}(T^*X)$. Then [Tra11, Theorem 3.3] also gives a lower bound for $\mathrm{vol}(L - F)$:

$$\mathrm{vol}(L - F) \geq L^n - nL^{n-1} \cdot Z(F) - n \sum_j (L + \nu_j u)^{n-1} \cdot \nu_j D_j.$$

In particular, if $\mathcal{O}_{TX}(1)$ is nef, then we can take $u = 0$ and we have

$$\mathrm{vol}(L - F) \geq L^n - nL^{n-1} \cdot F.$$

Our next result shows that L can be any pseudo-effective line bundle, which is just Theorem 1.2.

Theorem 3.1. Let X be a smooth projective variety of dimension n , and let L, M be two pseudo-effective line bundles with M movable. Then we have

$$\mathrm{vol}(L - M) \geq \mathrm{vol}(L) - n\langle L^{n-1} \rangle \cdot M.$$

Proof. This follows from Theorem 1.1, Proposition 1.1 (see below for the proof) and [BFJ09, Theorem A]. \square

Remark 3.3. When M is nef and L is pseudo-effective, Theorem 3.1 can be proved by using the singular Morse inequalities for line bundles (see [Bon98]). Without loss of generality, we can assume L is big and M is ample. Let $\omega \in c_1(M)$ be a Kähler metric. For any Kähler current $T \in c_1(L)$ with analytic singularities, $T - \omega$ is an almost positive curvature current of $L - M$ with analytic singularities. With the elementary pointwise inequality

$$\mathbf{1}_{X(\alpha - \beta, \leq 1)}(\alpha - \beta)^n \geq \alpha^n - n\alpha^{n-1} \wedge \beta$$

for positive $(1, 1)$ -forms, Theorem 3.1 then follows easily from [Bon98].

3.3 Proposition 1.1

Towards the transcendental version of Theorem 3.1, we give the proof of Proposition 1.1 which is essentially already known in [BFJ09].

Proof. It is obvious $(2) \Rightarrow (1)$. We will show that $(1) \Rightarrow (3) \Rightarrow (2)$, this then proves the equivalence of the above statements.

Firstly, we prove (3) \Rightarrow (2). To prove (2), we only need to consider the case when $\langle \alpha^n \rangle - n\langle \alpha^{n-1} \rangle \cdot \beta > 0$. By Theorem 1.1 we know $\alpha - \beta$ is big, thus (3) implies the volume function vol is \mathcal{C}^1 differentiable at the points $\alpha - t\beta$ for $t \in [0, 1]$. And we have

$$\left. \frac{d}{dt} \right|_{t=t_0} \text{vol}(\alpha - t\beta) = -n\langle (\alpha - t_0\beta)^{n-1} \rangle \cdot \beta.$$

This implies

$$\text{vol}(\alpha - \beta) = \text{vol}(\alpha) - \int_0^1 n\langle (\alpha - t\beta)^{n-1} \rangle \cdot \beta dt.$$

By Corollary 3.2 we have the inequality $\langle (\alpha - t\beta)^{n-1} \rangle \cdot \beta \leq \langle \alpha^{n-1} \rangle \cdot \beta$, then we get

$$\text{vol}(\alpha - \beta) \geq \text{vol}(\alpha) - n\langle \alpha^{n-1} \rangle \cdot \beta.$$

Next, the implication (1) \Rightarrow (3) is essentially [BFJ09, Section 3.2]. For reader's convenience, we briefly recall and repeat the arguments of [BFJ09, Section 3.2]. By [BFJ09, Corollary 3.4] (or the proof of [BDPP13, Theorem 4.1]), (1) implies

$$\text{vol}(\beta + t\gamma) \geq \beta^n + tn\beta^{n-1} \cdot \gamma - Ct^2$$

for an arbitrary nef class β , an arbitrary $(1, 1)$ -class γ and $t \in [0, 1]$. Here the constant C depends only on the class β, γ ; more precisely, the constant C depends on the volume of a big and nef class ω such that $\omega - \beta$ is pseudo-effective and $\omega \pm \gamma$ is nef.

Now take a log-resolution $\mu^*\alpha = \beta + [E]$, then we have

$$\begin{aligned} \text{vol}(\alpha + t\gamma) &\geq \text{vol}(\beta + t\mu^*\gamma) \\ &\geq \beta^n + tn\beta^{n-1} \cdot \mu^*\gamma - Ct^2 \\ &= \beta^n + tn\mu_*(\beta^{n-1}) \cdot \gamma - Ct^2. \end{aligned}$$

Note that the constant C does not depend on the resolution μ , since $\mu^*\omega - \beta$ is pseudo-effective and $\mu^*\omega \pm \mu^*\gamma$ is nef if ω has similar property with respect to α, γ . And we have $\text{vol}(\mu^*\omega) = \text{vol}(\omega)$. By taking limits of some sequence of log-resolutions, we get

$$\text{vol}(\alpha + t\gamma) \geq \text{vol}(\alpha) + tn\langle \alpha^{n-1} \rangle \cdot \gamma - Ct^2.$$

Replace γ by $-\gamma$, we then get

$$\text{vol}(\alpha) \geq \text{vol}(\alpha + t\gamma) - tn\langle (\alpha + t\gamma)^{n-1} \rangle \cdot \gamma - Ct^2.$$

Since α is big, by the concavity of movable intersections (see e.g. [BDPP13, Theorem 3.5]) we have

$$\lim_{t \rightarrow 0} \langle (\alpha + t\gamma)^{n-1} \rangle = \langle \alpha^{n-1} \rangle.$$

Then (3) follows easily from the above inequalities. \square

Remark 3.4. It is observed in [Den15] that the \mathcal{C}^1 differentiability of the volume function for transcendental $(1, 1)$ -classes holds on compact Kähler surfaces. And it is used to construct the Okounkov bodies of transcendental $(1, 1)$ -classes over compact Kähler surfaces.

3.4 Theorem 1.3

Finally, inspired by the method in our previous work [Xia13] (see also [Chi13]), we show Theorem 1.1 gives a Morse-type bigness criterion of the difference of two movable $(n-1, n-1)$ -classes, thus giving the proof of Theorem 1.3.

Proof. Denote the Kähler cone of X by \mathcal{K} , and denote the cone generated by cohomology classes represented by positive $(n-1, n-1)$ -currents by \mathcal{N} . Then by the numerical characterization of Kähler cone of [DP04] (see also [BDPP13, Theorem 2.1]) we have the cone duality relation

$$\overline{\mathcal{K}}^\vee = \mathcal{N}.$$

Without loss of generality, we can assume α, β are big. Then the existence of a strictly positive $(n-1, n-1)$ -current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$ is equivalent to the existence of some positive constant $\delta > 0$ such that

$$\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle \succeq \delta \langle \beta^{n-1} \rangle,$$

or equivalently,

$$\langle \alpha^{n-1} \rangle \succeq (1 + \delta) \langle \beta^{n-1} \rangle.$$

Here we denote $\gamma \succeq \eta$ if $\gamma - \eta$ contains a positive current.

In the following, we will argue by contradiction. By the cone duality relation $\overline{\mathcal{K}}^\vee = \mathcal{N}$, the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$ does not contain any strictly positive $(n-1, n-1)$ -current is then equivalent to the statement: for any $\epsilon > 0$ there exists some non-zero class $N_\epsilon \in \overline{\mathcal{K}}$ such that

$$\langle \alpha^{n-1} \rangle \cdot N_\epsilon \leq (1 + \epsilon) \langle \beta^{n-1} \rangle \cdot N_\epsilon.$$

On the other hand, we claim Theorem 1.1 implies

$$n(N \cdot \langle \alpha^{n-1} \rangle)(\alpha \cdot \langle \beta^{n-1} \rangle) \geq \langle \alpha^n \rangle (N \cdot \langle \beta^{n-1} \rangle)$$

for any nef $(1, 1)$ -class N . First note that both sides of the above inequality are of the same degree of each cohomology class. After scaling, we can assume

$$\alpha \cdot \langle \beta^{n-1} \rangle = N \cdot \langle \beta^{n-1} \rangle.$$

Then we need to prove $nN \cdot \langle \alpha^{n-1} \rangle \geq \langle \alpha^n \rangle$. Otherwise, we have $nN \cdot \langle \alpha^{n-1} \rangle < \langle \alpha^n \rangle$. And Theorem 1.1 implies there must exist a Kähler current in the class $\alpha - N$. Then we must have

$$\langle \beta^{n-1} \rangle \cdot (\alpha - N) > 0,$$

which contradicts with our scaling equality $\langle \beta^{n-1} \rangle \cdot (\alpha - N) = 0$.

Let $N = N_\epsilon$, we get

$$\begin{aligned} (1 + \epsilon)n(N_\epsilon \cdot \langle \beta^{n-1} \rangle)(\alpha \cdot \langle \beta^{n-1} \rangle) &\geq n(N_\epsilon \cdot \langle \alpha^{n-1} \rangle)(\alpha \cdot \langle \beta^{n-1} \rangle) \\ &\geq \langle \alpha^n \rangle (N_\epsilon \cdot \langle \beta^{n-1} \rangle). \end{aligned}$$

This implies

$$(1 + \epsilon)n\alpha \cdot \langle \beta^{n-1} \rangle \geq \langle \alpha^n \rangle.$$

Since $\epsilon > 0$ is arbitrary, this contradicts with our assumption $\langle \alpha^n \rangle - n\alpha \cdot \langle \beta^{n-1} \rangle > 0$. Thus there must exist a strictly positive $(n-1, n-1)$ -current in the class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$. \square

Remark 3.5. Let X be a smooth projective variety of dimension n and let $\text{Mov}_1(X)$ be the closure of the cone generated by movable curve classes. In [LX15], we show that any interior point of $\text{Mov}_1(X)$ is the form $\langle L^{n-1} \rangle$ for a unique big and movable divisor class. And under Demailly's conjecture on transcendental Morse inequality, this also extends to transcendental movable $(n-1, n-1)$ -classes over compact Kähler manifold. In particular, this extends to compact hyperkähler manifolds.

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